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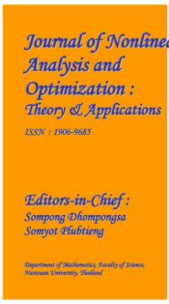

GRAPH THEORY AND ITS APPLICATIONS ARE SUBJECT OF SOME CONTRIBUTIONS

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ABSTRACT
Mathematical branch of graph theory has become a useful tool with many applications in different domains. Graph theory, which was first created to address issues with networks of connected nodes and edges, has grown into a rich and varied field of study that has made a substantial contribution to our knowledge of and ability to analyse complex systems in both theoretical and applied contexts. Graph theory has grown significantly since its inception in the 18th century, mostly due to developments in computer science, operations research, biology, chemistry, and other fields. The basic idea of a graph, which consists of edges (connections) and vertices (nodes), offers a flexible framework for representing structures and relationships in a variety of situations. Graphs are useful tools for representing interrelated phenomena in a variety of contexts, including social networks, transportation systems, computer networks, and biochemical interactions. They enable researchers to better understand, analyse, and optimise complicated systems in practical applications.
Keywords: intricate systems, graph theory, mathematics, networks, and related nodes andedges

INTRODUCTION

Mathematical branch of graph theory has become a useful tool with many applications in different domains. [1] Graph theory, which was first created to address issues with networks of connected nodes and edges, has grown into a rich and varied field of study that has made a substantial contribution to our knowledge of and ability to analyse complex systems in both theoretical and applied contexts. Graph theory has grown significantly since its inception in the 18th century, mostly due to developments in computer science, operations research, biology, chemistry, and other fields. [2] The basic idea of a graph, which consists of edges (connections) and vertices (nodes), offers a flexible framework for representing structures and relationships in a variety of situations. Graphs are useful tools for representing interrelated phenomena in a variety of contexts, including social networks, transportation systems, computer networks, and biochemical interactions. They enable researchers to better understand, analyse, and optimise complicated systems in practical applications.[3]

We examine the importance of graph theory and its numerous applications in a variety of domains in this paper. We explore the fundamental ideas of graph theory, emphasising its key characteristics, techniques, and algorithms. Additionally, we look at particular graph theory applications across a range of fields, showing how graph-theoretical methods have transformed problem-solving and decision-making in disciplines including optimisation, data mining, network analysis, and more.

This presentation seeks to emphasise the significance of graph theory as a mathematical discipline in contemporary research and practice by providing an outline of the field and its applications. [4] We cordially invite readers to go with us through the intriguing field of graph theory, which offers insights and answers to challenging real-world issues by bridging the gap between abstract ideas and concrete issues.

Basic Ideas and Qualities:

The study of graphs, which are mathematical structures used to represent pairwise relations between things, is the focus of the mathematical field of graph theory. A number of basic ideas and characteristics in the field of graph theory provide the framework for comprehending and evaluating different types of graph structures.[5]

An arrangement of vertices (nodes) and edges (links) connecting pairs of vertices is called a graph. Edges show the links or interconnections between the discrete items that are represented by vertices. Different graph classifications, such as directed and undirected graphs, are defined by the kinds of edges and how they are oriented.

In graph theory, connectedness and connectivity are crucial characteristics that establish whether pathways exist between vertices. If there is a path connecting each pair of vertices in a graph, then the graph is said to be linked. The ability of a graph to retain connections even after certain vertices or edges have been removed is referred to as connectivity.

A graph's paths and cycles are patterns of vertices and edges that move through it and reveal information about its connectivity and structure. A vertex's degree indicates how many edges are incident to it, indicating how significant it is in the graph.[6]

In a graph, the connectedness between vertices and edges is determined by adjacency and incidence relationships. Graphs are computationally represented using a variety of techniques, including adjacency lists and adjacency matrices, which enable effective algorithms and analyses.

It is possible to compare several graphs to see if they are structurally identical thanks to graph isomorphism. Finding patterns and structures in complicated graph systems requires the use of subgraphs, which are subsets of vertices and edges within a larger graph.[7] Certain graph types, like trees, bipartite graphs, and complete graphs, have special qualities that make them useful in a variety of applications. These foundational ideas and characteristics provide as the basis for investigating the complex and varied subject of graph theory.

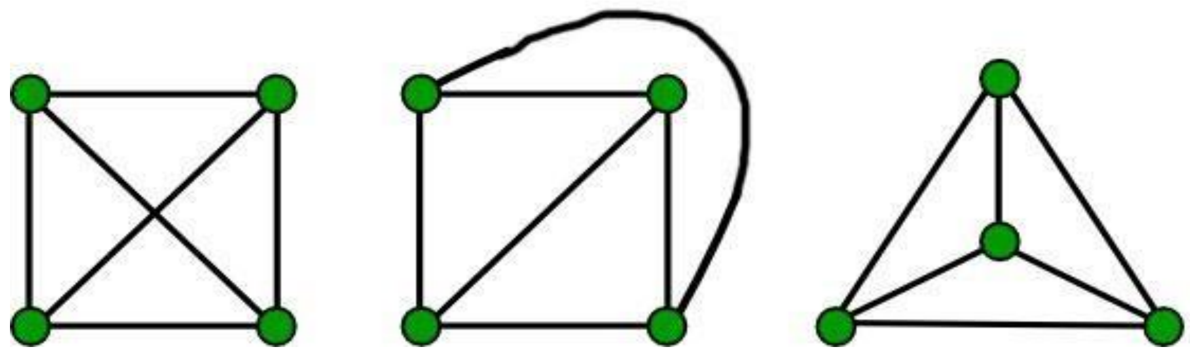
Planar Graph

Any kind of graph that can be drawn on a plane with edges that only overlap at their ends is called a planar graph. Stated otherwise, it is a graph whose edges do not cross over one another when it is contained in a plane. [8]Planar graphs are distinguished from non- planar graphs by this attribute; the latter require crossings in order to be drawn in a legal way.

According to formal definitions, a graph is planar if all of its edges can be drawn without crossing one another. We refer to this illustration as a planar embedding. A graph is considered planar if it admits at least one planar embedding. Because of their many uses and complex mathematical structure, planar graphs are studied in great detail in graph theory and related topics. They also feature a number of fascinating qualities.[9]

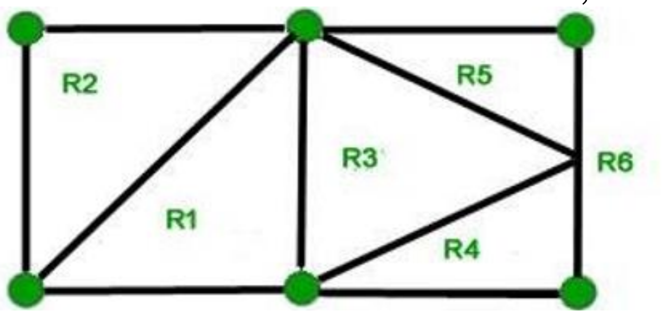
If a graph can be plotted on a plane without any edges crossing, it is said to be planar. The term "planar representation of the graph" refers to such a drawing. Even though a graph may be drawn with crossings, it may still be planar if the graph can be drawn in a different way without crossings.

Take the entire graph, for instance, with its two potential planar representations:



Areas within Linear Graphs

A graph's planar representation divides the plane into regions. With the exception of one unbounded zone, all of these regions are enclosed by edges. Take a look at the followinggraph, for instance.



Six zones in all, five bounded and one unbounded, are present. The number of places in which a graph's planar representations divide the plane is the same. [10] Euler discovered how the number of vertices and edges in a planar graph related to the number of regions in the graph.

Features of Linear Graphs:

No Cutting Edges: Any drawing of a planar graph on a plane will only contain edge intersections at their ends. All other sites of intersection are absent.

Regions: Often referred to as faces, the plane that contains a planar graph is separated into regions. Every face in the graph has a cycle enclosing it; one of these faces is commonly referred to as the "outer" face, encircling every other region.[11]

Euler's formula, which links a planar graph's number of vertices (V), edges (E), and faces (F) as V - E + F = 2, is one of the basic characteristics of planar graphs. For linked planargraphs with no loops or more than one edge, this formula is valid.

Embedding: A planar graph can have more than one embedding, or depiction, on a plane, and in terms of edge crossings and regions, each one will maintain the same general structure.

Circuit Design: To ensure that no wires cross and to produce a compact layout for printed circuit boards, connections between electronic components are represented using planar graphs in circuit design.

Map Colouring: The four-color theorem, which asserts that any planar map can be coloured with just four colours so that no two neighbouring sections have the same colour, is strongly connected to planar graphs. Applications of this theorem include colouring challenges for geographical maps and cartography.[12]

Network Routing: Planar graphs are used to simulate communication networks in network design and routing techniques. This helps to optimise data packet routing by preventing inter- node interference.

Algorithms for Drawing Graphs: A key idea in graph drawing techniques is the use of planar graphs to create visually appealing and useful representations of intricate networks.

Theorem Euler's Formula

n – e + r = 2 holds for a connected planar graph G with n vertices, e edges, and r region.

Proof

Using induction one, the number of edges in G, we demonstrate the theorem.

The foundation for induction G can only have one vertex if e = 0. that is, one infinite region(r = 1) and n = 1.

n-e+ r = 1 - 0 + 1 = 2 is the result.

The number of vertices of G is either 1 or 2, the first choice when the edge is a loop, if e = 1 (but it is not required). [13]As seen in the following Figure, these two alternatives result in two regions and one region, respectively.

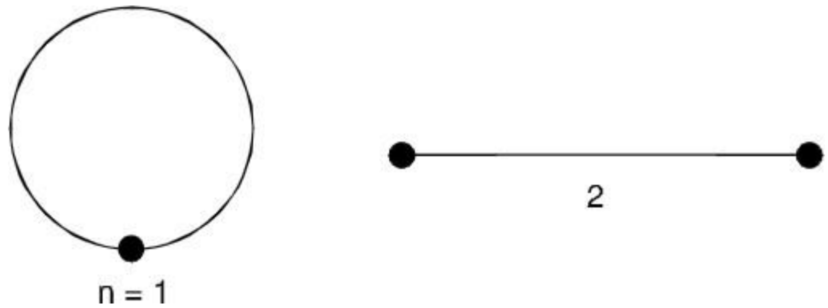


Fig 1.1: One-edge connected planar graphs

If there is no loop, then $n - e + r = 2 - 1 + 1 = 2$, and if there is a loop, then $n - e + r = 1 - 1 + 2 = 2$.

Thus, the outcome is accurate.

Induction hypothesis:

Currently, we are assuming that any linked plane graph G with $e - 1$ edges will provide the same conclusion.

Step of induction:

To create a linked super graph of G, we add one new edge, K, to G. This super graphis represented as G + K.

The following three options are available.

(i) If K is a loop, then the number of vertices stays the same but a new region enclosed by theloop is generated.

(ii) If K connects two different vertices of G, then one of G's regions splits into two, increasing the number of regions by 1, while leaving the number of vertices unaltered.

(iii) If K has just one vertex from G, then an additional vertex must be added, increasing thetotal number of vertices by one while maintaining the same number of regions.

If the number of vertices, edges, and regions in G is represented by n', e', and r', and the same is represented by n, e, and r in G + K. Next

$n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$ in case (i).

For example, $n - e + r = n' - (e' + 1) + (r' + 1) = n' - e' + r'$ in case (ii).$n - e + r = (n' + 1) - (e' + 1) + r' = n' - e' + r'$ in example (iii).

However, $n' - e' + r' = 2$ according to our induction hypothesis.Thus, $n - e + r = 2$ in each scenario.

For some connected graph G with $e - 1$ edges and a new edge K, every plane connectedgraph with e edges is now of the type G + K.

Therefore, the formula is true for all flat graphs via mathematical induction.

Corollary (1)

$n - e + r = K + 1$ if a plane graph has K components.

Applying Euler's formula to each component independently and keeping in mind not to count the infinite region more than once yields the following result.

Corollary (2)

The second corollary states that $e \leq 3n - 6$ if G is a connected simple planar graph with n (≥ 3)vertices and e edges.

Proof

Since the graphs being studied here are simple graphs, no multiple edges that could form regions of degree 2 or loops that could produce regions of degree 1 are allowed.) Each region is surrounded by at least three edges, and each edge belongs to exactly two regions.

$2e \geq 3r$ This and Euler's formula, $n - e + r = 2$, when combined, give $3r = 6 - 3n + 3e \leq 2e$, or$e \leq 3n - 6$.

Corollary (3):

$e \leq 2n - 4$ if G is a connected simple planar graph with n (≥ 3) vertices, e edges, and nocircuits of length 3.

Proof

Each region has at least four degrees if the graph is planar. Therefore, there are at least 4r edges total surrounding each region.

Since each edge has two neighbouring regions, there are 2e edges altogether surrounding all the regions. Therefore, we found that $2e \geq 4r$, which is the same as $2r \leq e$.
This and Euler's formula, $n - e + r = 2$, when combined, give $2r = 4 - 2n + 2e \leq e$, or $e \leq 2n - 4$.

Theorem

Two times the number of edges in the matching graph is the sum of the degrees of allthe regions in a map.[14]

Proof:

A map can be represented as a graph, with adjacent parts connected by edges and map regions represented by vertices. The number of neighbouring regions in a region on a map called its degree. As a result, the degree of the associated vertices in a graph equals the degree of a region on a map. We are aware that a graph's total degree of all its vertices equals twice its total number of edges.
Thus, the $2e = \Sigma \deg(R_i)$

Theorem

Let G be a basic connected planar (p, q)-graph with a minimum of K edges at each region's border. $(k - 2)q$ therefore $\leq k(p - 2)$.

Evidence

Each edge on G's boundary is contained within the bounds of precisely two regions. Furthermore, G might contain some pendent edges that are not inside any region's boundaries.
As a result, the total length of G's limits is less than the total number of its edges.For example, $kr \leq 2q$...(1)
However, since G is a linked graph, Euler's formula applies.$r = 2 + q - p$ is what we have…..(2)
When we replace (2) in (1), we have $k(2 + q - p) \leq 2q$, which implies $(k - 2)q \leq k(p - 2)$.

CONCLUSIONS

Graph theory offers a strong foundation for modelling and analysing complicated systems, making it an essential tool in many domains. Driven by technological developments and interdisciplinary collaborations, graph theory has consistently progressed from its modest beginnings in the 18th century to its current state as a thriving study. [15]The fundamental ideas and characteristics of graphs serve as the cornerstone for comprehending and addressing practical issues, and specific applications across a range of industries highlight the adaptability and potency of graph-theoretical methods. Graph theory continues to be a useful tool for obtaining understanding, generating predictions, and streamlining procedures as scientists and engineers delve further into the social sciences, sciences, and other fields. Researchers can find hidden patterns, pinpoint important linkages, and create effective solutions to challenging problems by utilising the power of graphs. Graph theory's transdisciplinary character promises further innovation and discovery in the future, reshaping the field of contemporary research and practice.

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